Design Sensitivity Analysis and Optimization of Nonlinear Structural Response Using Incremental Procedure

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This paper describes procedures for design sensitivity analysis of nonlinear structural response using the incremental solution scheme. The total and updated Lagrangian approaches are discussed. Sensitivity analyses for the stress, strain, and displacement constraints are developed. In the optimization process, the buckling load constraint is also imposed. Numerical aspects of design sensitivity analysis are discussed. Three numerical examples are solved to illustrate the design sensitivity analysis and optimization process. The numerical procedures are stable and work well. They can be applied to more complex structures to optimize them.

I. Introduction

IN numerical methods of optimization, one must determine the effect of a change in the current design on the cost and constraint functions, i.e., evaluate the gradients of response quantities with respect to design variables. This is commonly known as design sensitivity analysis and can constitute a major task in any structural optimization program. The gradients are also important in their own right as they respresent trend for the structural performance or constraint functions.

The derivatives of the cost and constraint functions are essential to compute a search direction in the optimization process. In general, there are two ways to compute the design derivatives. A straightforward way is to use the finite-difference approximation. For example,

$$\frac{\mathrm{d}g}{\mathrm{d}b_i} \simeq \frac{g[U(b_i+h),b_i+h]-g[U(b_i),b_i]}{h} \tag{1}$$

where g is a cost or constraint function, U is a displacement vector, b_i is a design variable, and h is a small perturbation in design. There are two serious shortcomings of using this approach for a nonlinear response problem. First, there is an uncertainty in the choice of perturbation, h. An improper choice may cause truncation or condition errors in the computation.[1] Second, when the number of design variables is large, there is an enormous increase in the number of nonlinear finite-element analyses. The procedure can, therefore, be prohibitively expensive.

The other way is to differentiate implicit functions analytically and develop expressions for the gradients. The design sensitivity analysis for linear structural analysis has been thoroughly investigated.[1-3] Until now only a few papers have been published about the design sensitivity analysis and optimization of nonlinear response. Khot,[4] and Khot and Kamat[5] derive an optimality criterion of uniform strain energy density to minimize the structure weight with a nonlinear buckling constraint. A recurrence relation is used to optimize truss structures. Kamat and Ruangsilasingha[6] propose a design sensitivity formulation for minimization of structural weight subjected to equality constraints for minimum potential energy as well as the requirement of singular Hessian of the potential energy. Ryu, Haririan, Wu, and Arora[7] have studied the adjoint variable and direct differentiation methods of design sensitivity analysis. They have compared the two approaches for linear as well as nonlinear problems. Haririan, Wu, and Arora[8] have implemented design sensitivity analysis in the ADINA program which uses an incremental solution scheme. Wu and Arora[9] have derived general procedures for design sensitivity analysis with geometric and material nonlinearities including nonlinear collapse load constraint. A general optimal structural design formulation imposing constraints on stresses, displacements, strains, as well as the nonlinear collapse load needs to be investigated. In this paper, such a general formulation is considered and procedures for design sensitivity analysis are discussed and illustrated with several examples.

One small difficulty with nonlinear analysis and design sensitivity analysis is the complexity of notation. Several formulations and notations are possible. We follow the notations used in Ref. 10.

II. Design Sensitivity Analysis: General Formulation

Structural optimization problem with nonlinear response can be described by the following general model:

$$\text{minimize } \psi_0(b,{}^tU) \tag{2}$$

$$\text{subject to } \psi_i(b,{}^tU) \le 0; \quad i=1, \text{NC} \tag{3}$$

and satisfying the equilibrium equation ${}^tQ(b,{}^tU)=0$ if the structure does not collapse, or ${}^cQ(b,{}^cU)\equiv{}^cR-{}^cF=0$ if the structure collapses with the constraints of Eq. (3) imposed at ${}^tU\equiv{}^cU$.

Here ψ_0 is a cost function, ψ_i is any constraint function not including the nonlinear buckling constraint, ${}^tU$ is an n-vector of nodal displacements, NC is the number of constraints, ${}^cR$ is the critical load vector, ${}^cF$ is the internal force vector at the critical load level, and the left superscript represents the load level. In the preceding equations, it is assumed that the design variables b are continuous and that all functions are continuously differentiable. The equation ${}^tQ(b,{}^tU)=0$ represents the nonlinear equilibrium equation for the finite-element model of the structural system subject to static loads. An explicit form of the equation is not known and it is not needed when the incremental/iterative solution scheme is used. If the structure collapses before the final load level is reached, the equilibrium equation can no longer be satisfied. Instead, the critical load vector is needed to impose the nonlinear buckling constraint. Note also that

Received May 21, 1986; revision received Nov. 16, 1986.

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the response ${}^tU$ at the final load level cannot be computed. Instead, the response of ${}^cU$ at the critical load level is calculated. Then the question is: How should constraints of Eq. (3) be imposed? This question will be investigated and numerical procedures will be described to implement the constraints.

The model is quite general since the design bound, stress, displacement, strain, and nonlinear buckling constraints can be included. Various constraints can be expressed as: stress, $|\sigma_i| - \sigma_a \leq 0$; displacement, $|{}^tU_i| - {}^tU_{i0} \leq 0$; strain, $|\epsilon_i| - \epsilon_{ia} \leq 0$; and buckling load, $1 - {}^cp \leq 0$; where σ_i is an effective stress at point i and σ_a is its limiting value, ${}^tU_i$ is the displacement at the ith point and ${}^tU_{i0}$ is its limiting value, ϵ_i is the effective strain, and ϵ_{ia} is its limiting value, ${}^cp$ is the buckling load factor. Detailed numerical procedures for calculating these quantities are discussed in Refs. 9–12.

The problem of design sensitivity analysis is to calculate gradient of a function ψ at given design $\boldsymbol{b}$ and calculated displacement ${}^t\boldsymbol{U}$ as (the arguments $\boldsymbol{b}, {}^t\boldsymbol{U}$ will be omitted for brevity):

$$\frac{d\psi}{d\boldsymbol{b}} = \frac{\partial\psi}{\partial\boldsymbol{b}} + \left(\frac{\partial\psi}{\partial^t\boldsymbol{U}}\right)\frac{d^t\boldsymbol{U}}{d\boldsymbol{b}} \tag{4}$$

The partial derivatives $\partial\psi/\partial\boldsymbol{b}$ and $\partial\psi/\partial^t\boldsymbol{U}$ are relatively easy to calculate since explicit dependence of ψ on $\boldsymbol{b}$ and ${}^t\boldsymbol{U}$ is usually known. It will be also seen later that $\partial\psi/\partial^t\boldsymbol{U}$ is quite easily obtained when incremental finite-element analysis is used. Calculation of the matrix $d^t\boldsymbol{U}/d\boldsymbol{b}$ is the major computational burden needing closer scrutiny. Two numerical procedures[1-3,7,13] have been used for linear structures: direct differentiation and adjoint variable methods. These methods can also be used for design sensitivity analysis of nonlinear structures. The criteria for choosing one over the other are the same as for the linear case.[2,7] Both methods need essentially the same quantities in numerical implementation. In the derivations to follow, only the direct differentiation method is used. It is assumed that the more efficient of the two procedures is used in actual numerical implementations.

In order to calculate $d^t\boldsymbol{U}/d\boldsymbol{b}$, the equilibrium equation for the structural system should be considered:

$${}^t\boldsymbol{Q}(\boldsymbol{b}, {}^t\boldsymbol{U}) \equiv {}^t\boldsymbol{R} - {}^t\boldsymbol{F} = 0 \tag{5}$$

where ${}^t\boldsymbol{R}$ is the externally applied nodal load vector which may be an explicit function of the design variables (it is assumed independent of displacements, although displacement dependent loads can be treated[9,10]), and ${}^t\boldsymbol{F}$ is the internal nodal force vector obtained from the calculated stress distribution.

Taking total derivatives of Eq. (5) with respect to design, we obtain

$$\frac{\partial^t\boldsymbol{Q}}{\partial\boldsymbol{b}} + \frac{\partial^t\boldsymbol{Q}}{\partial^t\boldsymbol{U}}\frac{d^t\boldsymbol{U}}{d\boldsymbol{b}} = 0; \quad \frac{\partial^t\boldsymbol{Q}}{\partial^t\boldsymbol{U}} = \frac{\partial^t\boldsymbol{R}}{\partial^t\boldsymbol{U}} - \frac{\partial^t\boldsymbol{F}}{\partial^t\boldsymbol{U}};$$

$$\frac{\partial^t\boldsymbol{Q}}{\partial\boldsymbol{b}} = \frac{\partial^t\boldsymbol{R}}{\partial\boldsymbol{b}} - \frac{\partial^t\boldsymbol{F}}{\partial\boldsymbol{b}} \tag{6}$$

It can be shown that $-\partial^t\boldsymbol{Q}/\partial^t\boldsymbol{U}$ is the tangential stiffness matrix ${}^t\boldsymbol{K}$ (Refs. 7, 10). The term $\partial^t\boldsymbol{R}/\partial^t\boldsymbol{U}$ is actually the load correction matrix (when the load depends on displacements), and is an unsymmetric contribution to the tangential stiffness matrix. If we assume that the external load is deformation independent and the associate flow rule is applicable for the material of the structure, then the tangential stiffness matrix is symmetric. Therefore, derivatives of the displacement vector can be computed if we can calculate $\partial^t\boldsymbol{Q}/\partial\boldsymbol{b}$ in Eq. (6). Note that this system of equations is linear even if the original system is nonlinear. Calculation of $\partial^t\boldsymbol{R}/\partial\boldsymbol{b}$ in Eq. (6) is quite straightforward, since explicit dependence of ${}^t\boldsymbol{R}$ on design is known. Partial derivatives of internal forces ${}^t\boldsymbol{F}$ with respect to the design variables need to be calculated.

III. Design Sensitivity Analysis: Numerical Procedures

Efficient numerical procedures of design sensitivity analysis depend on the procedures used for nonlinear structural analysis. Therefore, some details of the nonlinear analysis are discussed before design sensitivity analysis is described.

A. Nonlinear Analysis

For proper nonlinear analysis, it is important to use a consistent continuum mechanics formulation. The starting point for such an analysis is the principle of virtual work. The difference between linear and nonlinear analyses is that the principle must be written in a deformed configuration for the latter case. For the linear case, the deformations and displacements are assumed to be infintesimal, so there is no difference between deformed and undeformed configurations. However, for the nonlinear case, the deformed configuration (which is not known) must be used. A load incrementation coupled with iterations is used to satisfy equilibrium at a given load level. These procedures are described in Refs. 9–12 and are only summarized here.

In the derivation of incremental finite-element procedures, it is assumed that the equilibrium configuration at the load level t is known and it is desired at the level $t+\Delta t$. To accomplish this, an incremental form of the principle of virtual work is obtained by introducing incremental decompositions for stresses, strains, and displacements. Then the usual finite-element approximation is introduced and a matrix form of the incremental equilibrium equation is obtained. Since linear approximations are used, the equilibrium at $t+\Delta t$ will not be exactly satisfied. Therefore, iterations within the load step are needed.

Based on the aforementioned procedure, two approaches for nonlinear analysis have been pursued. The first one relates all the static and kinematic variables to the initial configuration, and is generally called total Lagrangian (TL) formulation. The second formulation, called the updated Lagrangian (UL), refers all the static and kinematic variables to an updated configuration. If consistent approximations are used, both formulations give the same final response. However, each formulation uses different stress and strain measures. The TL formulation uses second Piola-Kirchhoff stress and Green-Lagrange strain tensors. The UL formulation uses Cauchy stress and infintesimal strain tensors. One advantage of the TL formulation is that the second Piola-Kirchhoff stress tensor is invariant under a rigid body rotation whereas the Cauchy stress tensor is not. If large strains are present in an inelastic analysis, the UL formulation has to use a spin invariant stress measure such as the Jaumann stress rate tensor. It is possible to transform one stress measure to the other one.

The design sensitivity analysis can be performed using either the TL or UL formulation.[7,9] In the following, both formulations will be summarized. The equations are given for a finite-element, and usual assembly procedures are used to obtain the equilibrium equation for the entire structure.

The incremental equilibrium equation for the UL formulation is

$$({}^t_t\boldsymbol{K}_L + {}^t_t\boldsymbol{K}_{NL})\boldsymbol{U} = {}^{t+\Delta t}_{\ \ t}\boldsymbol{R} - {}^t_t\boldsymbol{F} \tag{7}$$

where $\boldsymbol{U}$ is the incremental displacement defined as $\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{U} - {}^t\boldsymbol{U}$, and

$${}^t_t\boldsymbol{K}_L = \int_{{}^tV} {}^t_t\boldsymbol{B}_L^T\, {}_t\boldsymbol{C}\, {}^t_t\boldsymbol{B}_L\, {}^t dV, \quad {}^t_t\boldsymbol{K}_{NL} = \int_{{}^tV} {}^t_t\boldsymbol{B}_{NL}^T\, {}^t\hat{\tau}\, {}^t_t\boldsymbol{B}_{NL}\, {}^t dV \tag{8}$$

${}^t_t\boldsymbol{K}_L$ is the linear strain incremental stiffness matrix, ${}_t\boldsymbol{C}$ is the tangential material property matrix measured at load level t, ${}^t_t\boldsymbol{K}_{NL}$ is the initial stress, geometric stiffness, or the nonlinear strain incremental stiffness matrix, V is the volume, ${}^t\hat{\tau}$ is the Cauchy stress matrix, ${}^t_t\boldsymbol{B}_L$ and ${}^t_t\boldsymbol{B}_{NL}$ are the linear and nonlinear incremental strain-displacement matrices at the load level t; the left superscript indicates the configuration in which the quantity is calculated, and the left subscript indicates the reference configuration. The second term on the right-hand side of Eq. (7) is the internal force vector given as

$$ {}^t_t\boldsymbol{F} = \int_{{}^tV} {}^t_t\boldsymbol{B}_L^T\, {}^t\tau\, {}^t\mathrm{d}V \tag{9} $$

where ${}^t\tau$ is the Cauchy stress vector (${}^t\tau_{11}$, ${}^t\tau_{22}$, ${}^t\tau_{33}$, ${}^t\tau_{12}$, ${}^t\tau_{13}$, ${}^t\tau_{23}$). ${}^{t+\Delta t}_{t}\boldsymbol{R}$ is the equivalent nodal force at level $(t+\Delta t)$ measured relative to configuration at t.

The incremental equilibrium equation for the TL formulation is

$$ ({}^t_0\boldsymbol{K}_L + {}^t_0\boldsymbol{K}_{NL})\boldsymbol{U} = {}^{t+\Delta t}_{0}\boldsymbol{R} - {}^t_0\boldsymbol{F} \tag{10} $$

where

$$ {}^t_0\boldsymbol{K}_L \int_{{}^tV} {}_0\boldsymbol{B}^t_{L0}\boldsymbol{C}^t_0\boldsymbol{B}^0_{\mathrm{L}}\mathrm{d}V, \quad {}^t_0\boldsymbol{K}_{NL} = \int_{{}^0V} {}_0\boldsymbol{B}^T_{NL}{}^t_0\hat{S}^t_0\boldsymbol{B}_{NL}{}^0\mathrm{d}V \tag{11} $$

${}^t_0\boldsymbol{K}_L$ is the linear incremental strain matrix, ${}_0\boldsymbol{C}$ is the incremental stress-strain material property matrix, ${}^t_0\boldsymbol{K}_{NL}$ is the initial stress, geometric stiffness, or the nonlinear incremental strain matrix, ${}^t_0\hat{S}$ is the matrix of 2nd Piola-Kirchhoff stresses, ${}^t_0\boldsymbol{B}_L$ and ${}^t_0\boldsymbol{B}_{NL}$ are the linear and nonlinear incremental strain-displacement matrices at load level t. The second term on the right-hand side of Eq. (10) is the internal force vector given as

$$ {}^t_0\boldsymbol{F} = \int_{{}^0V} {}^t_0\boldsymbol{B}^T_L\, {}^t_0\boldsymbol{S}\, {}^0\mathrm{d}V \tag{12} $$

where ${}^t_0\boldsymbol{S}$ is the second Piola-Kirchhoff stress vector (${}^t_0S_{11}$, ${}^t_0S_{22}$, ${}^t_0S_{33}$, ${}^t_0S_{12}$, ${}^t_0S_{13}$, ${}^t_0S_{23}$). ${}^{t+\Delta t}_{0}\boldsymbol{R}$ is the equivalent nodal force at $t+\Delta t$ measured relative to the 0 configuration.

Buckling Load Calculation

Many procedures have been implemented to compute the critical load.[11] One way to calculate the maximum load which the structure can carry safely is simply to perform an incremental analysis using the nonlinear formulation. The method is quite simple and effective, but can lead to high solution cost. In addition, the tangential stiffness matrix computed at the maximum load is ill-conditioned, so the gradient information cannot be computed. Procedures for calculating buckling load and its design derivative are fully discussed in Ref. 14, so they will not be included here.

B. Gradient Calculations

To complete the calculation of gradients in Eqs. (4) and (6), we need to calculate $\partial\psi/\partial b$, $\partial\psi/\partial U$, and $\partial {}^tQ/\partial b$. The main quantities needed to accomplish these calculations are the internal force partial derivatives with respect to state and design variables. The partial derivative with respect to the state variable is quite easy to calculate using the incremental equation:

$$ \delta {}^t\boldsymbol{F} = {}^t\boldsymbol{K}(\boldsymbol{b}, {}^t\boldsymbol{U})\delta {}^t\boldsymbol{U} \tag{13} $$

where $\delta {}^t\boldsymbol{F}$ is the increment in ${}^t\boldsymbol{F}$ corresponding to an increment ${}^t\boldsymbol{U}$, ${}^t\boldsymbol{K}$ is the tangential stiffness matrix, and $\boldsymbol{b}$ is a given design which is fixed during the analysis phase. Note that when left subscript is omitted the quantities are referred to both TL and UL formulations. The tangential stiffness matrix ${}^t\boldsymbol{K}$ for the UL or TL formulation can be identified from Eqs. (7) or Eq. (10) respectively as

$$ {}^t\boldsymbol{K} = {}^t_t\boldsymbol{K}_L + {}^t_t\boldsymbol{K}_{NL} \qquad {}^t\boldsymbol{K} = {}^t_0\boldsymbol{K}_L + {}^t_0\boldsymbol{K}_{NL} \tag{14} $$

Therefore, $\partial {}^t\boldsymbol{F}/\partial {}^t\boldsymbol{U} = {}^t\boldsymbol{K}$ which does not require any additional calculation.

There are a couple of ways to calculate partial derivatives of internal forces with respect to design variables. One way is to first calculate the total internal force as

$$ {}^t\boldsymbol{F} = \int_0^{{}^tU} {}^\tau\boldsymbol{K}(\boldsymbol{b}, {}^\tau\boldsymbol{U})\mathrm{d}^\tau\boldsymbol{U} = {}^t\boldsymbol{K}_s(\boldsymbol{b}, {}^t\boldsymbol{U}){}^t\boldsymbol{U} \tag{15} $$

where ${}^t\boldsymbol{K}_s$ is the secant stiffness matrix which relates total displacements to the total internal forces. A relationship between the tangential matrix ${}^t\boldsymbol{K}$ and the secant matrix ${}^t\boldsymbol{K}_s$ can be obtained by taking first order variation of Eq. (15) with respect to ${}^t\boldsymbol{U}$ and identifying

$$ {}^t\boldsymbol{K} = \frac{\partial({}^t\boldsymbol{K}_s{}^t\boldsymbol{U})}{\partial {}^t\boldsymbol{U}} + {}^t\boldsymbol{K}_s \tag{16} $$

where the given design $\boldsymbol{b}$ is held fixed. Also from Eq. (15), $\partial {}^t\boldsymbol{F}/\partial \boldsymbol{b}$ is

$$ \frac{\partial {}^t\boldsymbol{F}}{\partial \boldsymbol{b}} = \frac{\partial}{\partial \boldsymbol{b}}\int_0^{{}^tU} {}^\tau\boldsymbol{K}(\boldsymbol{b}, {}^\tau\boldsymbol{U})\mathrm{d}^\tau\boldsymbol{U} = \frac{\partial}{\partial \boldsymbol{b}}[{}^t\boldsymbol{K}_s(\boldsymbol{b}, {}^t\boldsymbol{U}){}^t\boldsymbol{U}] \tag{17} $$

Thus calculation of the total internal force in Eq. (15) or its design derivative in Eq. (10) needs integration over the entire displacement history.

The design derivative of element internal forces in the TL and UL formulations can also be obtained by taking partial derivatives of Eqs. (12) and (9), respectively, with respect to design variables as

$$ \frac{\partial}{\partial \boldsymbol{b}}({}^t_0\boldsymbol{F}) = \frac{\partial}{\partial \boldsymbol{b}}\int_{{}^0V} {}^t_0\boldsymbol{B}^T_L\, {}^t_0\boldsymbol{S}\, {}^0\mathrm{d}V \tag{18a} $$

$$ \frac{\partial}{\partial \boldsymbol{b}}({}^t_t\boldsymbol{F}) = \frac{\partial}{\partial \boldsymbol{b}}\int_{{}^tV} {}^t_t\boldsymbol{B}^T_L\, {}^t\tau\, {}^t\mathrm{d}V = \frac{\partial}{\partial \boldsymbol{b}}\int_{{}^0V} {}^t_t\boldsymbol{B}^T_L\, {}^t\tau\, {}^t_0X\, {}^0\mathrm{d}V \tag{18b} $$

where ${}^t_0\boldsymbol{S}$ and ${}^t\tau$ are the stress vectors, ${}^t_0\boldsymbol{B}_L$ and ${}^t_t\boldsymbol{B}_L$ are the incremental strain-displacement matrices defined earlier and ${}^t_0X$ is the determinant of the element deformation gradient which is given as ${}^0\rho/{}^t\rho$ and ρ is the mass density.[10] It is convenient to perform calculations in Eqs. (18a) and (18b) with the isoparametric element formulation. In such a formulation the integrand is expressed in terms of the element natural or intrinsic coordinates, r, s, and t and $\mathrm{d}V$ is replaced by a hypercube $\mathrm{d}r\mathrm{d}s\mathrm{d}t$ multiplied by the determinant J of Jacobian of the transformation for independent variables. Now, since the hypercube, $\mathrm{d}r\mathrm{d}s\mathrm{d}t$ is not a function of design, Eqs. (18) of the TL formulation can be differentiated directly and expressed as

$$ \frac{\partial}{\partial \boldsymbol{b}}({}^t_0\boldsymbol{F}) = \int \frac{\partial}{\partial \boldsymbol{b}}({}^t_0\boldsymbol{B}^T_L\, {}^t_0\boldsymbol{S})J\mathrm{d}r\mathrm{d}s\mathrm{d}t + \int {}^t_0\boldsymbol{B}^T_L\, {}^t_0\boldsymbol{S}\frac{\partial J}{\partial \boldsymbol{b}}\mathrm{d}r\mathrm{d}s\mathrm{d}t \tag{19} $$

Analogously, Eq. (18) for the UL formulation gives

$$ \frac{\partial}{\partial \boldsymbol{b}}({}^t_t\boldsymbol{F}) = \int \frac{\partial}{\partial \boldsymbol{b}}({}^t_t\boldsymbol{B}^T_L\, {}^t\tau)\, {}^t_0XJ\mathrm{d}r\mathrm{d}s\mathrm{d}t + \int {}^t_t\boldsymbol{B}^T_L\, {}^t\tau\frac{\partial}{\partial \boldsymbol{b}}({}^t_0XJ)\mathrm{d}r\mathrm{d}s\mathrm{d}t \tag{20} $$

For design with fixed domain, determinant of the deformation gradient ${}^t_0X$ does not explicitly depend on design variables. For trusses, membranes, and shear panels, the strain displacement matrices ${}^t_t\boldsymbol{B}_L$ and ${}^t_0\boldsymbol{B}_L$, and stresses ${}^t_0S$ and ${}^t\tau$ do not explicitly depend on cross-sectional properties and, therefore, design variables. For bending-type elements (e.g., beam and plate), the strain displacement matrix and stresses depend explicitly on the element cross-sectional geometry which must be accounted for in derivative calculations. Note that calculations in Eqs. (17) or (19) and (20) are essentially the same for the UL and TL formulations, respectively. In Eq. (17) integration over the volume is carried out first to calculate stiffness matrices, whereas in Eqs. (19) and (20) integration over the displacement history is carried out first to calculate the total stress. Note that partial derivatives of internal forces with respect to design require secant quantities. Thus all derivatives cannot be calculated using the tangent quantities used in the analysis phase.

Stress and Strain Derivatives

The stress or strain constraint functions are generally expressed as explicit functions of stress or strain components. Let $\boldsymbol{\sigma}$ be the stress vector and $\boldsymbol{e}$ be the strain vector. In the TL or UL formulation, the stress and strain vectors $\boldsymbol{\sigma}$ and $\boldsymbol{e}$ are given as ${}^t_0S$ and ${}^t_0\epsilon$, or ${}^t\tau$ and ${}^t_t\epsilon$, respectively. The design sensitivities of the stress and strain constraints are then given as

$$\frac{\mathrm{d}\psi^\sigma}{\mathrm{d}\boldsymbol{b}}=\frac{\mathrm{d}\psi^\sigma}{\mathrm{d}\boldsymbol{\sigma}}\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\boldsymbol{e}}\left(\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{b}}+\frac{\partial \boldsymbol{e}}{\partial {}^t\boldsymbol{U}}\frac{\mathrm{d}{}^t\boldsymbol{U}}{\mathrm{d}\boldsymbol{b}}\right) \tag{21a}$$

$$\frac{\mathrm{d}\psi^\epsilon}{\mathrm{d}\boldsymbol{b}}=\frac{\mathrm{d}\psi^\epsilon}{\mathrm{d}\boldsymbol{e}}\left(\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{b}}+\frac{\partial \boldsymbol{e}}{\partial {}^t\boldsymbol{U}}\frac{\mathrm{d}{}^t\boldsymbol{U}}{\mathrm{d}\boldsymbol{b}}\right) \tag{21b}$$

where ψ^σ and ψ^ϵ denote stress and strain constraint functions, respectively, and the stress is assumed only related to the element strain. In Eqs. (21a) and (21b) three quantities in the stress and strain derivatives, $\partial\boldsymbol{e}/\partial\boldsymbol{b}$, $\partial\boldsymbol{e}/\partial{}^t\boldsymbol{U}$, and $\mathrm{d}\boldsymbol{\sigma}/\mathrm{d}\boldsymbol{e}$ need to be calculated for computation of sensitivities for stress and strain constraints. The calculation of the three quantities is dependent on the finite-element formulation used (TL or UL).

The design derivative of element strains cannot be obtained directly from the incremental form of strain-displacement relations. Accordingly, if the elastic material is considered, the basic equation of Green-Lagrange strain or Almansi strain should be used for the derivation. The element strain-displacement relation for the TL formulation is given as[10]

$${}^t_0\epsilon_{ij}=½({}^t_0u_{i,j}+{}^t_0u_{j,i}+{}^t_0u_{k,i}{}^t_0u_{k,j}) \tag{22}$$

where ${}^t_0\epsilon_{ij}$ is the Green-Lagrange strain tensor. For the UL formulation the strain-displacement relation is given as[10]

$${}^t_t\epsilon_{ij}=½({}^t_tu_{i,j}+{}^t_tu_{j,i}-{}^t_tu_{k,i}{}^t_tu_{k,j}) \tag{23}$$

where ${}^t_t\epsilon_{ij}$ is the Almansi strain tensor. Here $\boldsymbol{u}$ is the displacement field and usual partial-differentiation notation is used.

The Green-Lagrange strains in the vector form are given as

$${}^t_0\epsilon=({}^t_0\boldsymbol{B}_{L0}+{}^t_0\boldsymbol{B}_{L1}){}^t\boldsymbol{U} \tag{24}$$

where the linear strain-displacement matrices ${}^t_0\boldsymbol{B}_{L0}$ and ${}^t_0\boldsymbol{B}_{L1}$ are easily defined using element shape functions.[9] The design derivative of the Green-Lagrange strain vector is obtained as

$$\frac{\partial {}^t_0\epsilon}{\partial \boldsymbol{b}}=\frac{\partial}{\partial \boldsymbol{b}}[({}^t_0\boldsymbol{B}_{L0}+{}^t_0\boldsymbol{B}_{L1}){}^t\boldsymbol{U}] \tag{25}$$

Similarly, the Almansi strains in the vector form can be written as

$${}^t_t\epsilon=({}^t_t\boldsymbol{B}_{L0}-{}^t_t\boldsymbol{B}_{L1}){}^t\boldsymbol{U} \tag{26}$$

where the linear strain displacement matrices ${}^t_t\boldsymbol{B}_{L0}$ and ${}^t_t\boldsymbol{B}_{L1}$ are defined in a manner similar to ${}^t_0\boldsymbol{B}_{L0}$ and ${}^t_0\boldsymbol{B}_{L1}$ except that derivatives of the shape functions are taken with respect to the deformed coordinates ${}^tx_i$. The design derivative of Almansi strain vector is obtained as

$$\frac{\partial {}^t_t\epsilon}{\partial \boldsymbol{b}}=\frac{\partial}{\partial \boldsymbol{b}}[({}^t_t\boldsymbol{B}_{L0}-{}^t_t\boldsymbol{B}_{L1}){}^t\boldsymbol{U}] \tag{27}$$

If inelastic material behavior is considered the total strain at load level t also depends on stress and strain histories. Then, Eqs. (24) and (26) cannot be used to compute total strains. In other words, the total strain, composed of elastic and plastic strains, cannot be expressed by a secant formula. As we have mentioned in Sec. III. B., the partial design derivative requires a secant formula which is an explicit function of design. Thus, the partial design derivative of strain for inelastic material behavior is difficult to compute. However, a finite-difference method may be used and this is discussed in Sec. IV.

The derivatives of strains with respect to the displacements can be obtained from the incremental strain-displacement relations. Since the strain derivative is computed at the final response ${}^t\boldsymbol{U}$, the displacement increment $\boldsymbol{U}$ is zero. Thus, the nonlinear incremental strain vanishes. Thus, the strain derivative for the TL and UL formulations are obtained as

$$\frac{\partial {}^t_0\epsilon}{\partial {}^t\boldsymbol{U}}={}^t_0\boldsymbol{B}_L \qquad \frac{\partial {}^t_t\epsilon}{\partial {}^t\boldsymbol{U}}={}^t_t\boldsymbol{B}_L \tag{28}$$

The total stress derivative $\mathrm{d}\boldsymbol{\sigma}/\mathrm{d}\boldsymbol{e}$ can be obtained from the incremental stress-strain relations. Accordingly, the total stress derivatives with respect to strain for TL and UL formulations can be obtained as

$$\frac{\mathrm{d}{}^t_0S}{\mathrm{d}{}^t_0\epsilon}={}_0\boldsymbol{C}\equiv[{}_0C_{ijk\ell}] \qquad \frac{\mathrm{d}{}^t\tau}{\mathrm{d}{}^t_t\epsilon}={}_t\boldsymbol{C}\equiv[{}_tC_{ijk\ell}] \tag{29}$$

where ${}_0\boldsymbol{C}$ and ${}_t\boldsymbol{C}$ are the incremental constitutive matrices in the TL and UL formulations, respectively.

C. Sensitivity of Buckling Constraint

The sensitivity of buckling constraint ($\psi^b\equiv 1-{}^cp\le 0$) is given as $-\mathrm{d}^cp/\mathrm{d}\boldsymbol{b}$ at given $\boldsymbol{b}$ and calculated ${}^c\boldsymbol{U}$. The critical displacement ${}^c\boldsymbol{U}$ corresponds to the critical load level ${}^c\boldsymbol{R}={}^cp\boldsymbol{R}$. Several ways to calculate sensitivity vector for the critical load are described in Refs. 9 and 14, so details are omitted here. The most effective numerical procedure is

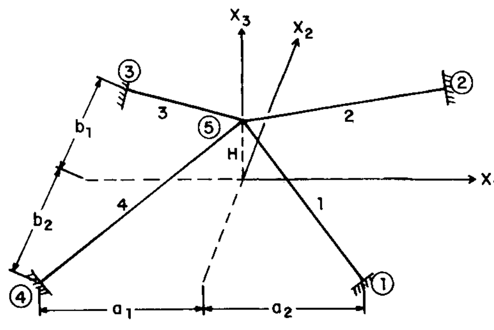

Fig. 1 Four bar shallow truss.

given by the formula:

$$\frac{\mathrm{d}^c p}{\mathrm{d}\boldsymbol{b}} = y^{\mathrm{T}} \left[\frac{\partial^c \boldsymbol{F}}{\partial \boldsymbol{b}} - {}^c p \, \frac{\mathrm{d}^t \boldsymbol{R}}{\mathrm{d}\boldsymbol{b}} \right] \Big/ (y^T \boldsymbol{R}) \tag{30}$$

where ${}^c\boldsymbol{F}$ is the internal force vector at the critical load, and y is eigenvector representing the buckled shape calculated from ${}^t\boldsymbol{K}y = 0$ with ${}^t\boldsymbol{K}$ as the tangential stiffness matrix calculated at the critical displacement ${}^c\boldsymbol{U}$. Note that design derivatives of internal forces at the critical load $\partial^c\boldsymbol{F}/\partial\boldsymbol{b}$ and external loads $\mathrm{d}^t\boldsymbol{R}/\mathrm{d}\boldsymbol{b}$ are also needed in the sensitivity analysis of other constraints [see Eq. (6)].

IV. Design Sensitivity Analysis by Semi-Analytical Approach

In design sensitivity analysis of nonlinear structural response, most of the needed quantities are already calculated in the analysis phase. Specifically, derivatives of internal forces, stresses and strains with respect to the state variables are available without any additional calculations when incremental solution scheme is used. Therefore, only the total design derivatives of state variables ($\mathrm{d}^t\boldsymbol{U}/\mathrm{d}\boldsymbol{b}$) and the partial design derivatives of stress $\partial\sigma/\partial\boldsymbol{b}$, strain $\partial\epsilon/\partial\boldsymbol{b}$, and internal force $\partial^t\boldsymbol{F}/\partial\boldsymbol{b}$ for each finite-element are needed to complete sensitivity analysis. The computation of $\mathrm{d}^t\boldsymbol{U}/\mathrm{d}\boldsymbol{b}$ requires only the solution of a linear system in Eq. (6). The design derivatives $\partial\sigma/\partial\boldsymbol{b}$, $\partial\epsilon/\partial\boldsymbol{b}$, and $\partial^t\boldsymbol{F}/\partial\boldsymbol{b}$, especially for bending-type elements, are finite-element related and require more tedious computation and programming effort. Thus it is suggested that a semi-analytical approach be used for the computation of partial design derivatives $\partial\sigma/\partial\boldsymbol{b}$, and $\partial\epsilon/\partial\boldsymbol{b}$, and $\partial^t\boldsymbol{F}/\partial\boldsymbol{b}$. In this approach, the original structure is analyzed; it is then modified by perturbing a design variable. Keeping the state variable constant, the partial design derivative of stress is given as

$$\frac{\partial\sigma}{\partial b_k} = \frac{\sigma[(b_k + h_k), {}^t U(b_k)] - \sigma[b_k, {}^t U(b_k)]}{h_k} \tag{31}$$

where h_k is a small perturbation in the kth design variable and ${}^t\boldsymbol{U}$ is the final displacement of original structure. Similarily, partial design derivative of strains and internal force vector are calculated. Since the computation of partial design derivative by finite-difference does not require reanalysis of the structure (state variables are fixed while the design is perturbed) the design sensitivity coefficients of the nonlinear response can be calculated efficiently.

V. Design Optimization with Multiple Constraints

In general, the structure can collapse before the final load level is reached during the incremental solution scheme. Thus, the equilibrium equation ${}^t\boldsymbol{R} - {}^t\boldsymbol{F} = 0$ cannot be used for the computation of the total design derivative of the state variable, $\mathrm{d}^t\boldsymbol{U}/\partial\boldsymbol{b}$. The design sensitivity information for other constraints except the nonlinear buckling constraint cannot be calculated. Therefore, design must be first improved to have a stable structure before other constraints can be imposed to optimize the structure. However, there are some difficulties in this approach. First, although the load carrying capacity can be improved up to the applied load level, the tangent stiffness matrix computed at the final load level becomes singular; i.e., when the critical load is equal to the final applied load. Second, one cannot ensure that the structure will not go into the unstable region again when other constraints are imposed. Thus, the design may oscillate between the stable and unstable regions during the iterative process and convergence will be difficult to achieve. An alternate approach is to optimize the structure by simultaneously imposing all the active constraints. That is, the nonlinear buckling constraint is imposed at the critical load level whereas other constraints are imposed at a modified response ${}^t\boldsymbol{U} = {}^c\boldsymbol{U} + \Delta^t\boldsymbol{U}$, where $\Delta^t\boldsymbol{U}$ is the additional response due to the residual force ${}^t\boldsymbol{R} - {}^c\boldsymbol{R}$ acting on the structure. Numerically, the additional response, $\Delta^t\boldsymbol{U}$ is impossible to calculate because the residual force is applied on an unstable structure. The tangent stiffness matrix is singular at the critical load level. However, some approximation to the incremental response $\Delta^t\boldsymbol{U}$ when added to the critical response ${}^c\boldsymbol{U}$ can avoid the above mentioned oscillations in the iterative design process. Two approaches are proposed to compute the additional response: 1) $\Delta^t\boldsymbol{U} = {}^\tau\boldsymbol{K}^{-1} \times ({}^t\boldsymbol{R} - {}^c\boldsymbol{R})$, where ${}^\tau\boldsymbol{K}$ is the tangential stiffness matrix computed at load level τ very close to the critical point; and 2) $\Delta^t\boldsymbol{U} = {}^\tau\boldsymbol{K}_L^{-1}({}^t\boldsymbol{R} - {}^c\boldsymbol{R})$, where ${}^\tau\boldsymbol{K}_L$ is the linear incremental strain matrix (linear part of tangent stiffness matrix). The geometric stiffness matrix, which tends to reduce the stiffness as deformation increases, is neglected. The first approach has two drawbacks. First, the additional response, $\Delta^t\boldsymbol{U}$ is very sensitive to the tangent stiffness matrix computed near the critical point and it is difficult to decide where the tangent stiffness matrix ${}^\tau\boldsymbol{K}$ should be calculated. Second, the tangent stiffness matrix ${}^\tau\boldsymbol{K}$ is computed near the critical point on the equilibrium path. Since the displacement is highly nonlinear near the critical point, it is computationally difficult to accurately find the response ${}^\tau\boldsymbol{U}$ near the critical load level from which the tangential matrix ${}^\tau\boldsymbol{K}$ can be calculated. Therefore in this paper, the second approach is used in numerical examples. Since the linear incremental strain matrix ${}^\tau\boldsymbol{K}_L$ is positive definite, the computation of the additional response is quite stable.

Note also that the tangential matrix ${}^t\boldsymbol{K}$ (or $-\partial^t\boldsymbol{Q}/\partial\boldsymbol{U}$), used for computing the design sensitivity of constraints in Eq. (6), is singular at the critical load level. Thus the modified tangential stiffness matrix, i.e., ${}^\tau\boldsymbol{K}_L$ is also used in the computation of total design derivative of state variables, $\mathrm{d}^t\boldsymbol{U}/\mathrm{d}\boldsymbol{b}$ when nonlinear buckling constraint is active.

VI. Example Problems

Using the derived design sensitivity expressions and the procedures, several numerical examples are solved. Geometric as well as material nonlinearities are treated. Constraints on stresses, displacements, strains and the buckling load are imposed simultaneously. All the examples involve only the axial force (truss) elements. A small program is developed to implement the UL formulation for analysis as well as design sensitivity analysis. The analysis part of the program was verified[9] by comparing results with the ADINA program.[15] The design sensitivity analysis with both geometric and material nonlinearities for stress, displacement, and collapse load was verified by a simple forward finite-difference scheme. Some of the results of verification are also given in Refs. 7-9. Problems with only the critical load constraint have been solved and discussed in other papers[9,14] and will not be included here. The formula for critical load sensitivity has been verified in Ref. 14 using an analytical example and finite-difference schemes.

Three example problems which consider the general constraints are discussed. The purpose of these examples is to show that stress/strain, displacement, collapse load, and design-variable bound constraints can be imposed simultaneously in the optimal design process with geometric and material nonlinearities. The design sensitivity expressions developed in the paper will be verified if converging optimal solutions are obtained. In addition, all the optimal designs given in the paper are new. Therefore they can be useful in future developments of the subject. All calculations are performed on a PRIME 750 computer running under PRIMOS 19.3. The optimal design problem is formulated as minimization of total volume (weight) of the structure subject to various constraints identified in the description of examples. The program IDESIGN[16] is employed in a black-box mode

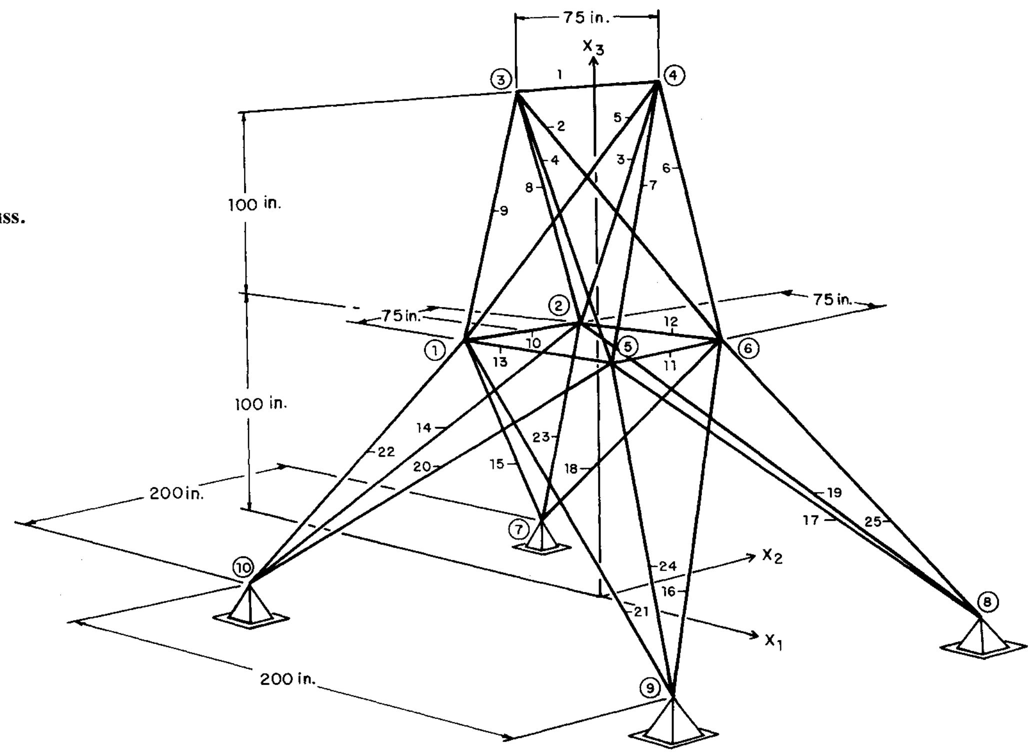

Fig. 2 Twenty-five bar space truss.

for design optimization. The option of sequential quadratic programming algorithm with Hessian updating and potential constraint strategy is used. Detailed design histories are given in Ref. 9. In discussing optimization results, the following notations are used:

NIT = number of iterations to reach optimum recognized by satisfaction of convergence criteria

NCF = number of calls for constraint function evaluation

NLS = number of load steps for nonlinear analysis

ACV = constraint feasibility check; maximum allowed violation of constraints at the optimum

ACS = convergence criteria; norm of the normalized direction vector or gradient of the Lagrangian should be less than ACS for accepting the optimum point

AAC = acceptable accuracy for the critical load; defined as ratio of norms of unbalanced load to the load increment

VCM = maximum constraint violation at the optimum

CP = value of convergence parameter at optimum

CPU = total CPU time in seconds

A. Four Bar Truss

The four-bar truss of Fig. 1 ($a_1 = 60$, $a_2 = 144$, $b_1 = 120$, $b_2 = 72$, $H = 3$ in.) is taken from Ref. 6. There the problem was formulated as maximization of compressive load at node 5 for a given volume of 1000 in.^{3}: this gave a load of 173.83697 lb. In the present paper, the problem is optimized to carry the same vertical load of 173.83697 lb in the negative x_3 direction and a horizontal load of 100.0 lb in the x_2 direction at node 5. The following three cases are considered: 1) NG1—only the buckling load constraint is imposed; 2) NG2—displacement limit of 1.2 in. is imposed at node 5 in each direction and a stress limit of 1500 psi is imposed in each member and; 3) NG3—same as case NG2 except that the displacement limit imposed at node 5 in x_3 direction is 0.8 in.

The three cases are also designed to prevent failure from snap-through buckling. The initial design is chosen as 1.734715 in.^{2} for each member.[6] The lower and upper bounds for each member are 1.0 and 10.0 in.^{2} respectively. The material used is aluminum with elastic modulus, $E = 10{,}000$ ksi.

The final results for each case are given in Table 1. After the program converged, the solutions were examined which revealed that the structure had collapsed at the first iteration for all cases. The procedure described in the paper was used to change the design to satisfy the constraints while increasing the load carrying capacity up to the applied load level. The optimal volume of case NG1 is slightly lower than that given in Ref. 6 where only the vertical load is considered. It shows that the additional horizontal load of 100 lb does not reduce the load carrying capacity of the structure. On the contrary, it appears to have stiffened the structure. The case NG3 has a more severe constraint on the displacement in the x_3 direction at node 5, compared to the case NG2. Therefore, as expected, the optimum cost function value for case NG3 is higher than that of case NG2. The final active constraints for the case NG2 include stress constraints in each member whereas for the case NG3 displacement in x_3 direction at node 5 is active. The cases NG2 and NG3 were also optimized with only linear response and the following solutions were obtained:

NG2: $b = (1.0, 1.0, 1.28428, 1.65954)$; Vol = 676.46 in.^{3}; NIT = 4. Active constraints: stress in numbers 3 and 4 and lower bound on b_1 and b_2.

NG3: $b = (1.0, 1.428, 1.31257, 1.85213)$; Vol = 725.1 in.^{3}; NIT = 17. Active constraints: displacement in x_3 direction at node 5 and lower bound on b_1.

For NG3, the volume reached optimum value at the 10th iteration, but precise convergence criterion was satisfied at the 17th iteration. In both the cases, the linear structure is about 35% lighter than the nonlinear structure and the final designs are quite different. The critical load constraint is not active. Thus, we can see that the structure softens when geometric nonlinearities are included. The optimal structure considering linear response would actually collapse under the applied loads.

B. Twenty-Five Bar Truss

A slightly larger 25-member space truss shown in Fig. 2 is formulated as an optimal design problem with geometric nonlinearity. The material considered is linearly elastic with modulus as $E = 10{,}000$ ksi. The initial design information and applied loads are given in Table 2. Note that the starting

Table 1 Optimum results for four bar truss

	Case (area, in.^{2})		
Element	NG1[a]	NG2[b]	NG3[c]
1	1.4395	2.1103	1.5155
2	2.8145	2.4520	3.2792
3	1.0000	1.8155	1.0000
4	2.0214	1.1587	2.4510
Optimum volume, in^3	997.0805	1022.760	1131.375

[a]NIT = 19, NCF = 57, ACV = 1.0E-5, ACS = 1.0E-4. VCM = 4.9360E-8, CP = 1.8821E-6, NLS = 20. AAC = 1.0E-9, CPU = 547. [b]NIT = 8, NCF = 9, ACV = 1.0E-5, ACS = 1.0E-4. VCM = 7.3882E-8, CP = 8.5129E-8, NLS = 20. AAC = 1.0E-9, CPU = 38. [c]NIT = 19, NCF = 47, ACV = 1.0E-5, ACS = 1.0E-4. VCM = 1.1892E-7, CP = 8.2069E-5, NLS = 20. AAC = 1.0E-9, CPU = 272.

design is selected so that the structure collapses under the unusually heavy load. Constraints on the element strains are imposed as follows: members 1-5 and 10-21, 0.02; and members 6-9 and 22-25, 0.004. The displacement constraints imposed at nodes 3 and 4 are: 30 in. in x_1 and x_2 directions, and 5 in. in the x_3 direction. Stress constraints are not imposed.

The optimum areas for the seven design groups shown in Table 2 are (4.2457, 32.0850, 65.7110, 0.25275, 10.7570, 31.3630, 19.5770) in.^{2} with a volume of 86,185 in.^{3} (NIT = 18, NCF = 25, ACV = 1.0E-5, ACS = 1.0E-4, VCM = 3.3726E-7, CP = 2.515E-5, NLS = 20, AAC = 1.0E-9, CPU = 897). The design iteration history was examined after the optimal solution was obtained. It revealed that the structure had indeed collapsed with the critical load factor of 0.551456 at the first iteration. The full load-carrying capacity was recovered immediately after the first iteration and convergence was obtained at the 18th iteration. At the optimum design, the strains in elements 1, 4, 6, 9, 10, 16, 18, 21, and 24 were at their limit values.

The structure was also optimized with linear response and the final design was as follows: $b = (6.76646, 16.73, 128.718, 1.5273, 2.27739, 13.8232, 48.5696)$; Vol = 102, 282 in.^{3}, NIT = 19; and active constraints: strains in members 6, 11, 16, 18, 24, 25. The critical load constraint was not active at the optimum. In this case, the nonlinear structure hardens and a more efficient design is obtained.

C. Nine Bar Plane Truss

The statically determinate truss, shown in Fig. 3, is optimized with constraints on displacement at each node and the strain in each element. The design variable and member numbers are also shown in Fig. 3. Loads on the structure are: 1200 kips in $-x_2$ direction at node 5 and 1600 kips in $-x_1$ direction at node 6. The constraint information is as follows: strain limits for members 1-3; 0.005; members 4, 8, 9; 0.01, members 5-7; 0.0075; displacement limits as 5 and 10 in. in the x_1 and x_2 directions, respectively, at all nodes. The structure is analyzed by considering geometric as well as material nonlinearities. The material is considered to be elastic-plastic with elastic modulus, $E = 200{,}000$ ksi, tangent modulus, $E_T = 50{,}000$ ksi and initial yielding stress, $\sigma_y = 100$ ksi. To investigate the role of the nonlinear buckling constraint in the optimal design process two cases are considered. In case 1 the starting design of smaller cross-sectional areas of 0.2 in.^{2} for each member is used. In this case the structure collapses before reaching the final load level at the first iteration. On the other hand, in case 2 the starting design of larger cross-sectional areas of 1.0 in.^{2} for each member is chosen. The upper and lower bounds on the design variables are 0.05 and 10 in.^{2}, respectively.

The final results for both the cases are summarized in Table 3. An examination of the detailed design histories[9] revealed that the structure collapsed at the first, third, and fourth iterations for case 1. For case 2, it did not collapse

Table 2 Design data for 25 bar space truss

	Initial Design			
Design variable, group no.	Elements	Initial design	Lower bound	Upper bound
1	1	0.2000E+01	0.2500	0.1000E+04
2	2,3,4,5	0.2000E+01	0.1000	0.1000E+04
3	6,7,8,9	0.2000E+01	0.5000	0.1000E+04
4	10,11,12,13	0.2000E+01	0.2500	0.1000E+04
5	14,15,16,17	0.2000E+01	0.1000	0.1000E+04
6	18,19,20,21	0.2000E+01	0.1000	0.1000E+04
7	22,23,24,25	0.2000E+01	0.5000	0.1000E+04

Applied load, kips			
Node	x_1	x_2	x_3
3	300	−4000	−3000
4	300	+4000	−3000

Table 3 Optimum results for nine bar plane truss

	Geometric and material nonlinearities	
Design variable, group no.	Case 1[a]	Case 2[b]
1	0.186230	0.186230
2	3.350100	3.350100
3	2.656000	2.656000
4	0.095745	0.095738
5	0.225600	0.225570
Optimum volume in.^{3}	240.3086	240.3087

[a]NIT = 26, NCF = 62, ACV = 1.0E-5, ACS = 1.0E-4. VCM = 4.9525E-7, CP = 4.4013E-5, NLS = 20. AAC = 1.0E-9, CPU = 1170. [b]NIT = 22, NCF = 66, ACV = 1.0E-5, ACS = 1.0E-4. VCM = 2.6287E-7, CP = 7.313E-6, NLS = 20. AAC = 1.0E-9, CPU = 1058.

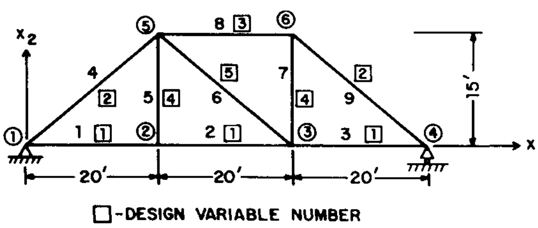

Fig. 3 Nine bar plane truss.

during the entire iterative process. Both the cases converge to the same optimum solution. The strain constraint for members 4 and 8 is active at the optimum. Using case 2 as an example, it can be concluded that for some structures optimum designs can be obtained without violating the nonlinear buckling constraint if the initial design and design bounds are selected properly. However, a general optimal design program should not have restrictions on the starting design. Besides, it is difficult to assure that the design will not get into unstable regions during the iterative process.

VII. Discussion and Conclusions

Design sensitivity analysis for nonlinear structures using incremental/iterative solution procedure is derived. In such a procedure, geometric as well as material nonlinearities can be treated quite consistently and routinely. Whereas nonlinear analysis of structures requires solution of nonlinear equations using load incrementation and iterations within it, the design sensitivity analysis requires the solution of linear equations. In addition, many of the quantities needed in sensitivity analysis are already calculated in the analysis phase. Specifically, derivatives of internal forces, stresses, and strains with respect to the state variables are obtained without any additional calculations when the incremental solution scheme is used. Thus design sensitivity analysis of nonlinear structures without shape variations requires only a fraction of the computational effort needed to calculate the nonlinear response. In fact, recent experience with optimal design of truss structures using ADINA[8] shows that most of the computational effort is spent in the analysis phase only. Thus, with a little additional computational effort, derivatives of response quantities can be calculated.

The preceding discussion also points out a need to make the entire analysis/optimization process more efficient for nonlinear structures. In this regard it is desirable to use sensitivity coefficients to predict the displacements for design at the next iteration. The displacements can be then improved using an iteration. Such a "mixed iteration" approach can be highly efficient for nonlinear structures. It has been evaluated on some small scale problems[9,17] and has shown considerable promise for large scale applications.

Design sensitivity analysis for the collapse load constraint is briefly discussed. A very simple and effective procedure has been discovered and evaluated.[14] It requires very little additional calculations to compute design gradient of the critical load. It is essential to impose the critical load constraint in optimal design of nonlinear structures. The structure can collapse before the full load level is reached in the incremental analysis procedure. Thus, the load carrying capacity of the structure must be improved before the optimal design process can proceed any further. However, numerical implementation shows the process to be unstable (oscillatory divergence) if other constraints are ignored while load carrying capability of the structure is improved. It is better to impose all the constraints by the suggested procedure even when the structure becomes unstable. Numerical convergence and stability of the process is considerably improved.

Based on the investigations presented, the following conclusions can be drawn:

1) Design sensitivity analysis of nonlinear response including the critical load is quite effective using the incremental structural analysis procedure. Geometric as well as material nonlinearities can be easily incorporated into the numerical procedure.

2) Optimization of nonlinear structures cannot be performed without the collapse load constraint.

3) Inclusion of geometric nonlinearities in the analysis and design optimization process can give substantially different designs. If the structure softens, the optimum design with linear response can fail catastrophically. On the other hand, if the structure hardens, inefficient optimum designs are obtained with linear response.

Acknowledgment

This research was initiated under a project sponsored by the National Science Foundation, USA, Grant CEE82-13851.

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